Statistik Bisnis

Week 14

Linear Regression

Learning Objectives

In this chapter, you learn:

- How to use regression analysis to predict the value of a dependent variable based on an independent variable
- The meaning of the regression coefficients b_0 and b_1
- How to evaluate the assumptions of regression analysis and know what to do if the assumptions are violated
- To make inferences about the slope and correlation coefficient
- To estimate mean values and predict individual values

Correlation vs. Regression

- A scatter plot can be used to show the relationship between two variables
- Correlation analysis is used to measure the strength of the association (linear relationship) between two variables
  - Correlation is only concerned with strength of the relationship
  - No causal effect is implied with correlation
  - Scatter plots were first presented in Ch. 2
  - Correlation was first presented in Ch. 3

Introduction to Regression Analysis

- Regression analysis is used to:
  - Predict the value of a dependent variable based on the value of at least one independent variable
  - Explain the impact of changes in an independent variable on the dependent variable

Dependent variable: the variable we wish to predict or explain

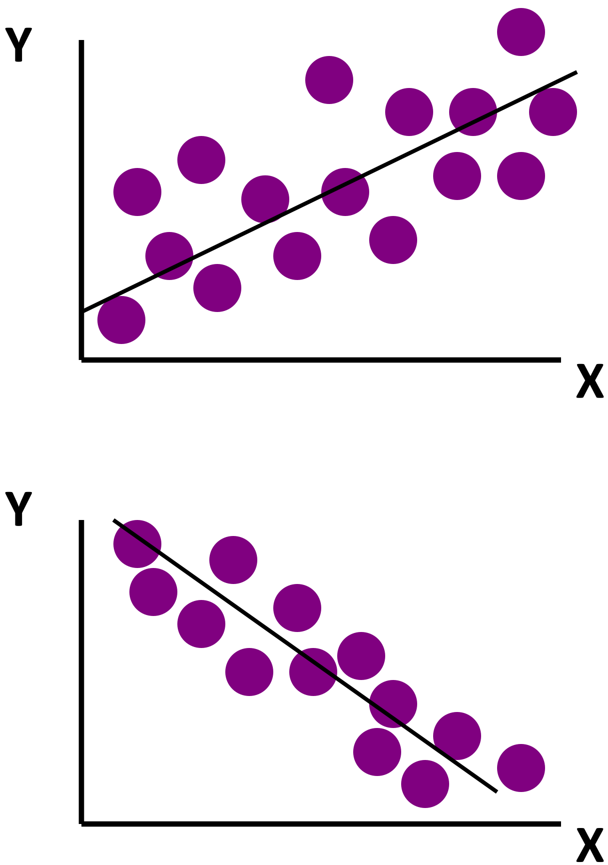
Independent variable: the variable used to predict or explain the dependent variable

Simple Linear Regression Model

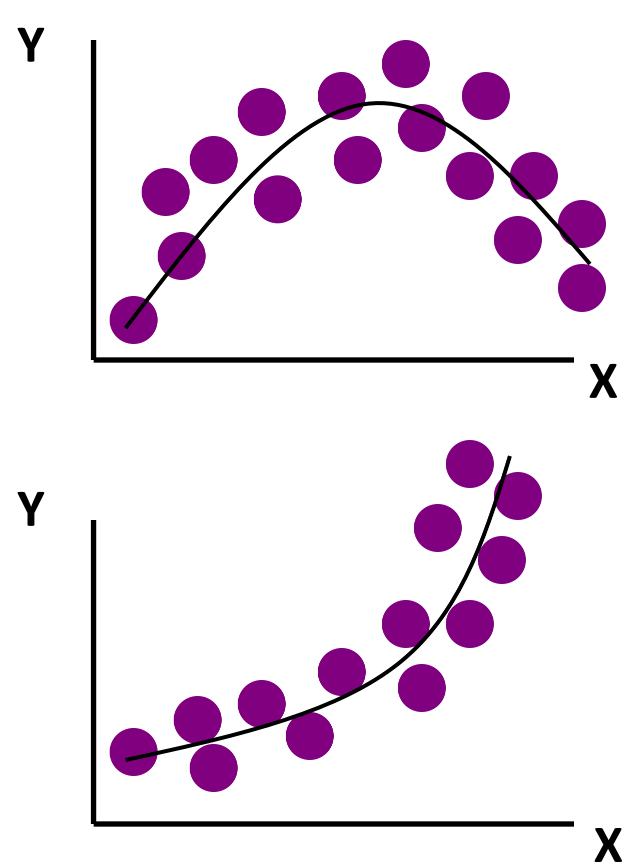
- Only **one** independent variable, X
- Relationship between X and Y is described by a linear function
- Changes in Y are assumed to be related to changes in X

Types of Relationships

Linear relationships

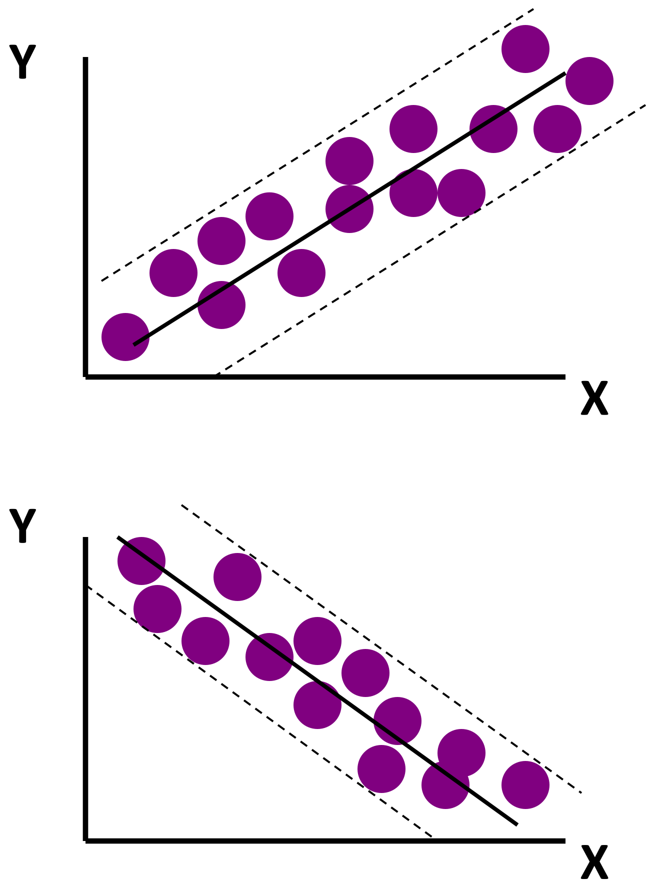


Curvilinear relationships

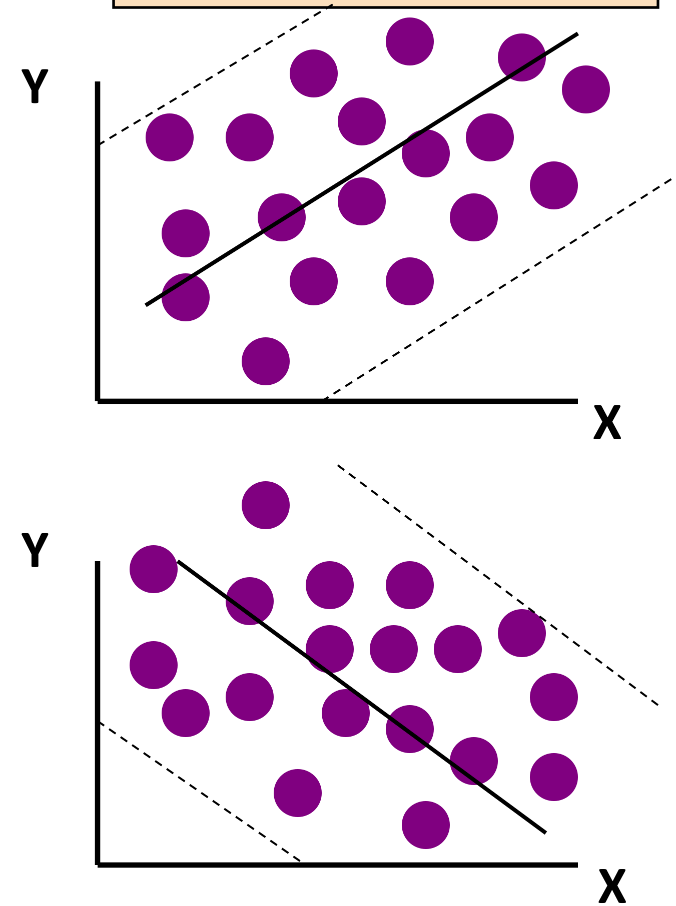


Types of Relationships

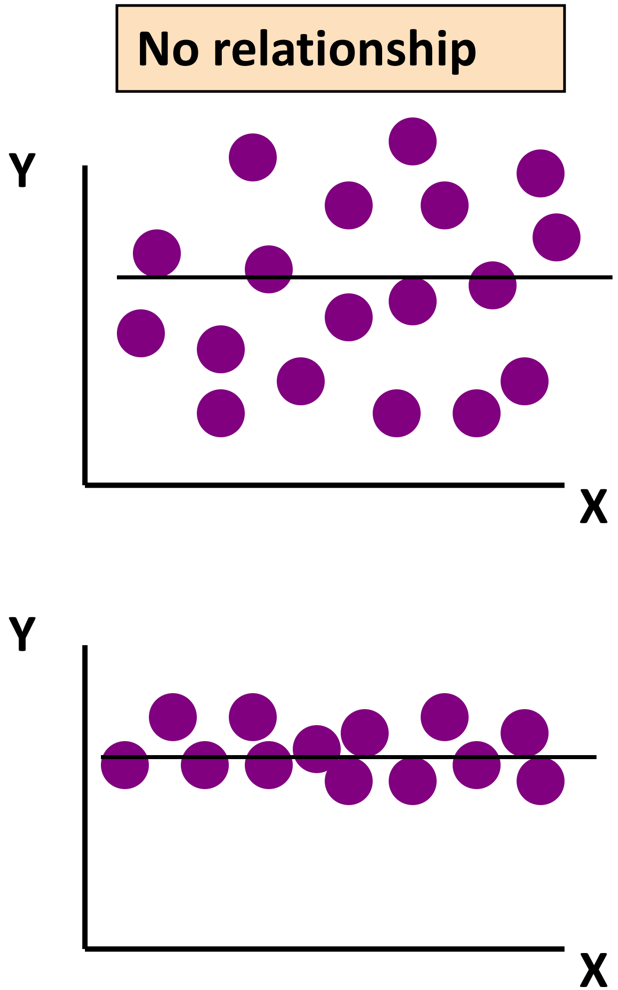
Strong relationships



Weak relationships



Types of Relationships



Simple Linear Regression Model

Population Y intercept

Population Slope Coefficient

Independent Variable

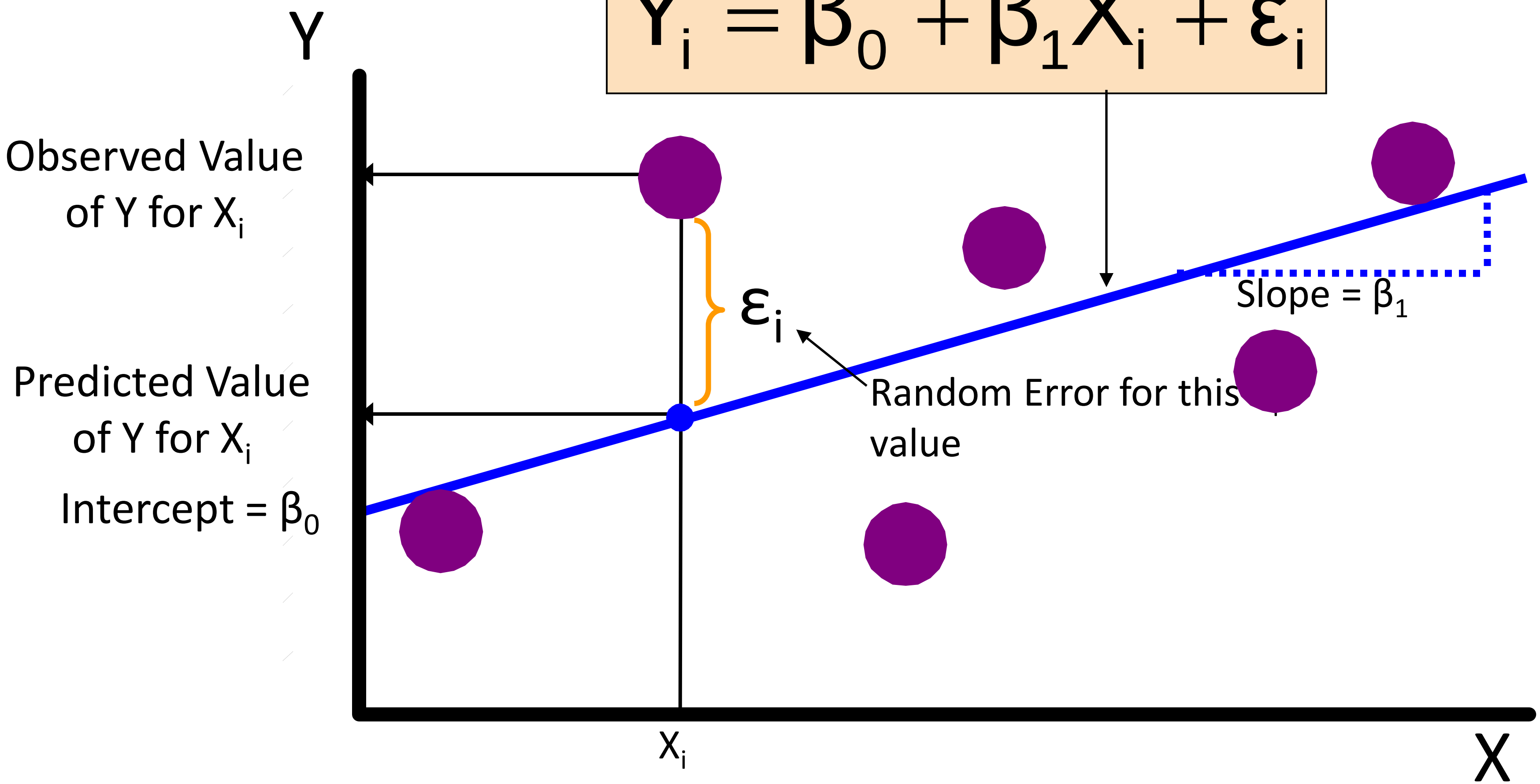
Random Error term

Dependent Variable

$$\boxed{Y_i = \underbrace{\beta_0 + \beta_1 X_i}_{\text{Linear component}} + \underbrace{\varepsilon_i}_{\text{Random Error component}}}$$

Simple Linear Regression Model

$$Y_i = \beta_0 + \beta_1 X_i + \varepsilon_i$$



Simple Linear Regression Equation (Prediction Line)

The simple linear regression equation provides an estimate of the population regression line

Estimated (or predicted) Y value for observation i

Estimate of the regression intercept

Estimate of the regression slope

Value of X for observation i

$$\hat{Y}_i = b_0 + b_1 X_i$$

The Least Squares Method

b_0 and b_1 are obtained by finding the values of that minimize the sum of the squared differences between Y and $\hat{Y}$:

$$\boxed{\min \sum (Y_i - \hat{Y}_i)^2 = \min \sum (Y_i - (b_0 + b_1 X_i))^2}$$

Linear Trend Model

$$\hat{Y}_i = b_0 + b_1 X_i$$

$$b_1 = \frac{\sum_{i=1}^{n} X_i Y_i - \frac{\left(\sum_{i=1}^{n} X_i\right)\left(\sum_{i=1}^{n} Y_i\right)}{n}}{\sum_{i=1}^{n} X_i^{\,2} - \frac{\left(\sum_{i=1}^{n} X_i\right)^2}{n}}$$

$$b_0 = \overline{Y} - b_1 \overline{X}$$

Interpretation of the Slope and the Intercept

- b_0 is the estimated mean value of Y when the value of X is zero

- b_1 is the estimated change in the mean value of Y as a result of a one-unit increase in X

Simple Linear Regression Example

- A real estate agent wishes to examine the relationship between the selling price of a home and its size (measured in square feet)

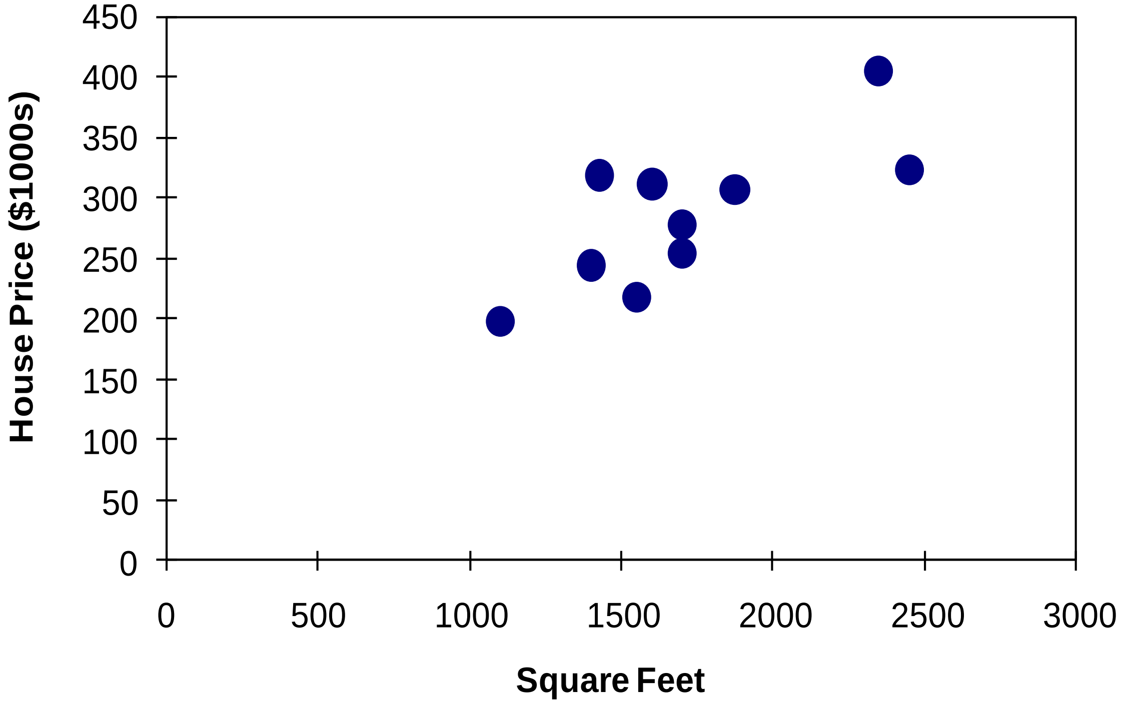
- A random sample of 10 houses is selected
  - Dependent variable (Y) = house price in $1000s
  - Independent variable (X) = square feet

Simple Linear Regression Example: Data

House Price in $1000s (Y)	Square Feet (X)
245	1400
312	1600
279	1700
308	1875
199	1100
219	1550
405	2350
324	2450
319	1425
255	1700

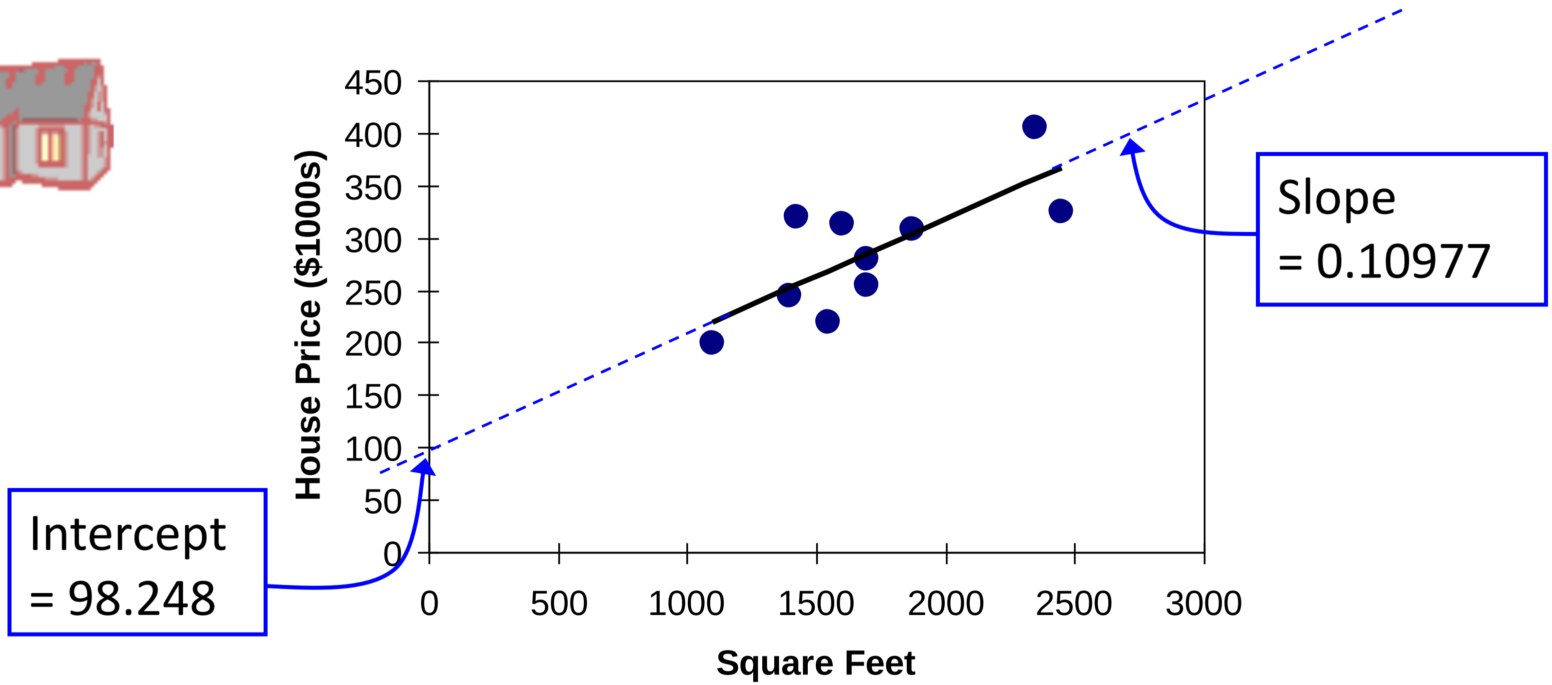
Simple Linear Regression Example: Scatter Plot

House price model: Scatter Plot



Simple Linear Regression Example: Graphical Representation

House price model: Scatter Plot and Prediction Line



$$\widehat{\text{house price}} = 98.24833 + 0.10977\,(\text{square feet})$$

Simple Linear Regression Example: Interpretation of b_o

$$\widehat{\text{house price}} = \boxed{98.24833} + 0.10977\,(\text{square feet})$$

- b_0 is the estimated mean value of Y when the value of X is zero (if X = 0 is in the range of observed X values)

- Because a house cannot have a square footage of 0, b_0 has no practical application

Simple Linear Regression Example: Interpreting b_1

$$\widehat{\text{house price}} = 98.24833 + \boxed{0.10977}\,(\text{square feet})$$

- b_1 estimates the change in the mean value of Y as a result of a one-unit increase in X
  - Here, b_1 = 0.10977 tells us that the mean value of a house increases by .10977($1000) = $109.77, on average, for each additional one square foot of size

Simple Linear Regression Example: Making Predictions

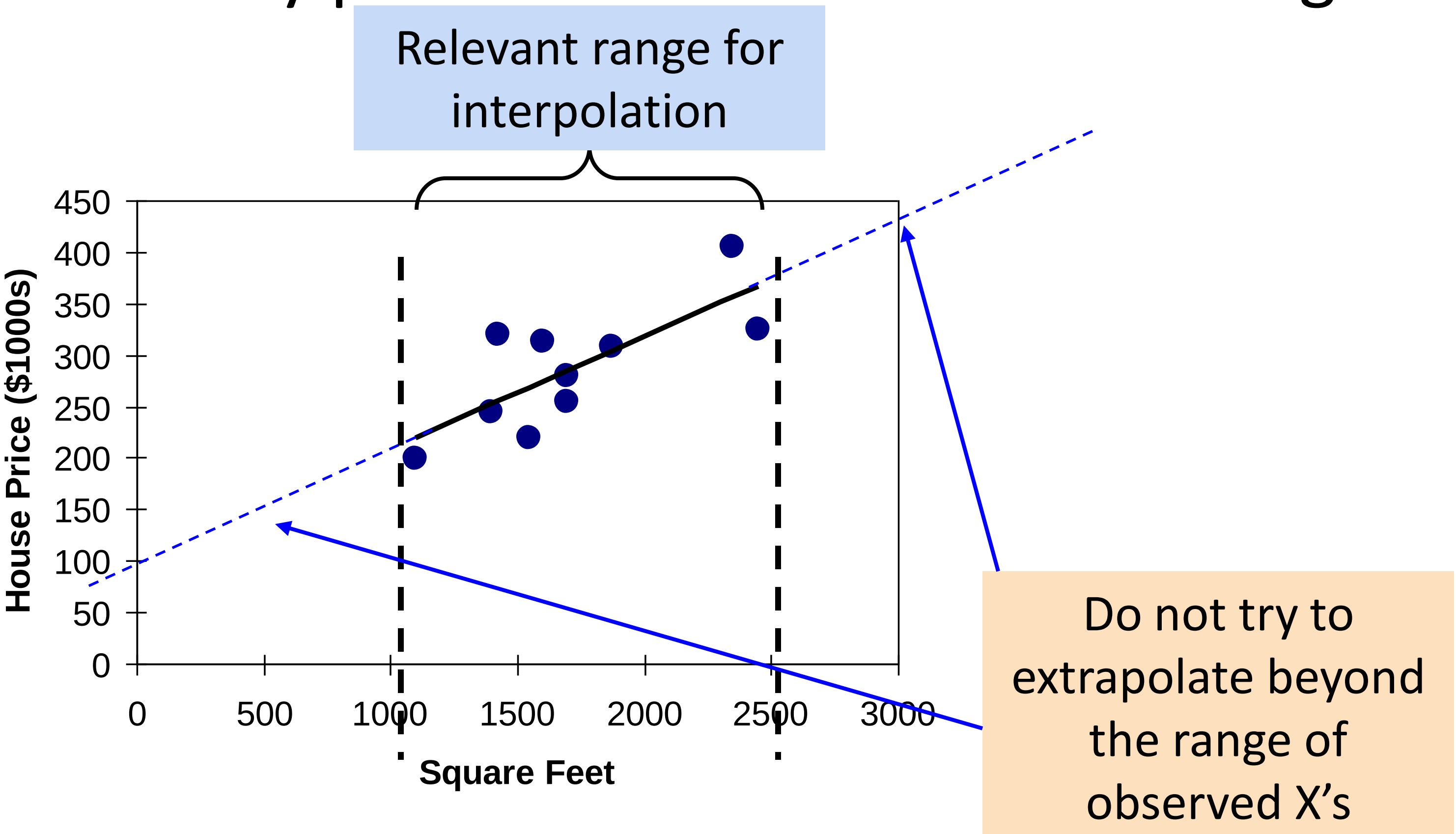
Predict the price for a house with 2000 square feet:

$$\widehat{\text{house price}} = 98.24833 + 0.10977\,(\text{sq.ft.})$$
$$= 98.24833 + 0.10977(2000)$$
$$= 317.78$$

The predicted price for a house with 2000 square feet is 317.78($1,000s) = $317,780

Simple Linear Regression Example: Making Predictions

- When using a regression model for prediction, only predict within the relevant range of data



Measures of Variation

- Total variation is made up of two parts:

$$SST \quad = \quad SSR \quad + \quad SSE$$

Total Sum of Squares	Regression Sum of Squares	Error Sum of Squares
$SST = \sum (Y_i - \overline{Y})^2$	$SSR = \sum (\hat{Y}_i - \overline{Y})^2$	$SSE = \sum (Y_i - \hat{Y}_i)^2$

where:

$\overline{Y}$ = Mean value of the dependent variable

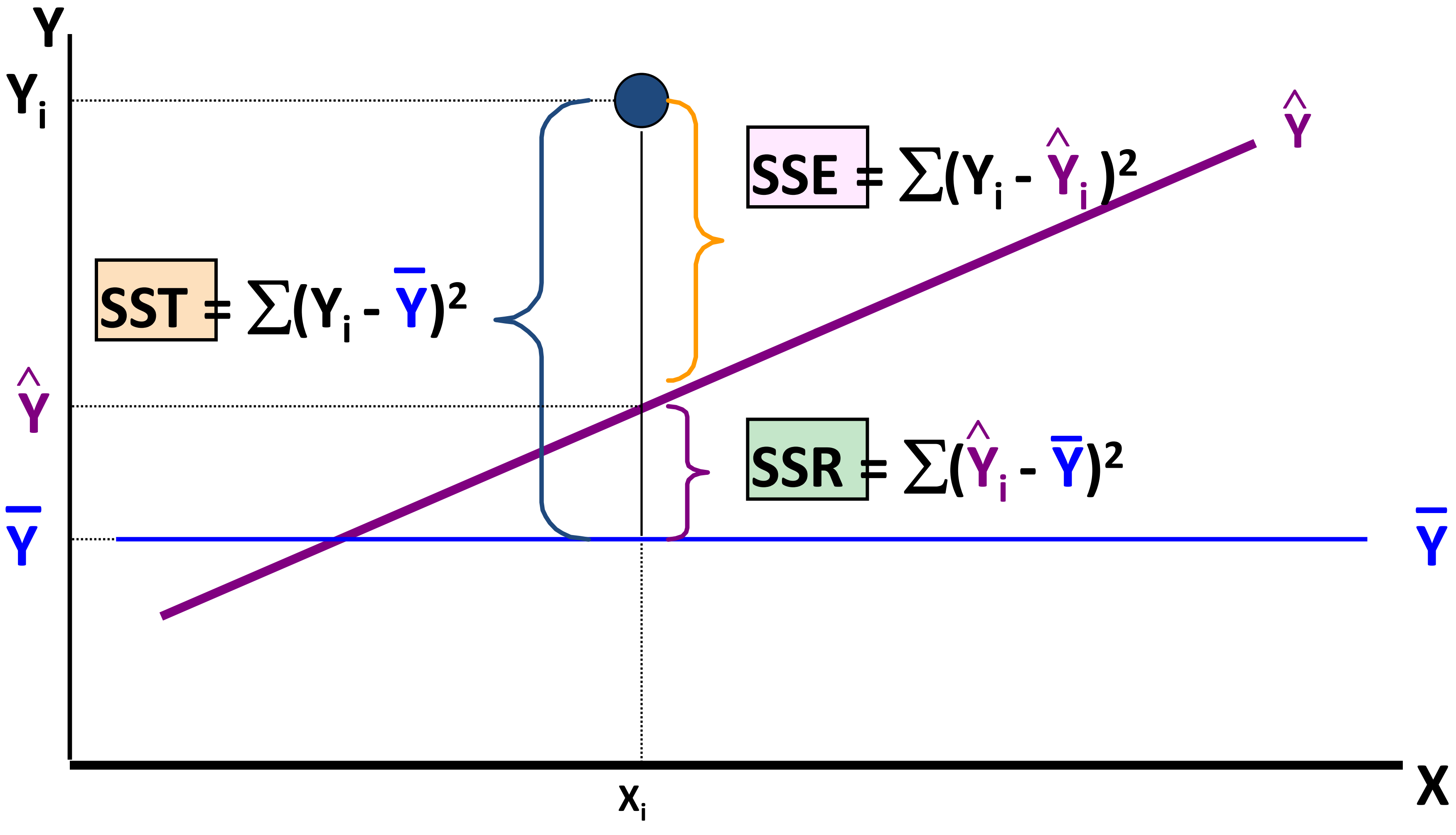
Y_i = Observed value of the dependent variable

= Predicted value of Y for the given X_i value

Measures of Variation

- SST = total sum of squares (Total Variation)
  - Measures the variation of the Y_i values around their mean $\bar{Y}$
- SSR = regression sum of squares (Explained Variation)
  - Variation attributable to the relationship between X and Y
- SSE = error sum of squares (Unexplained Variation)
  - Variation in Y attributable to factors other than X

Measures of Variation



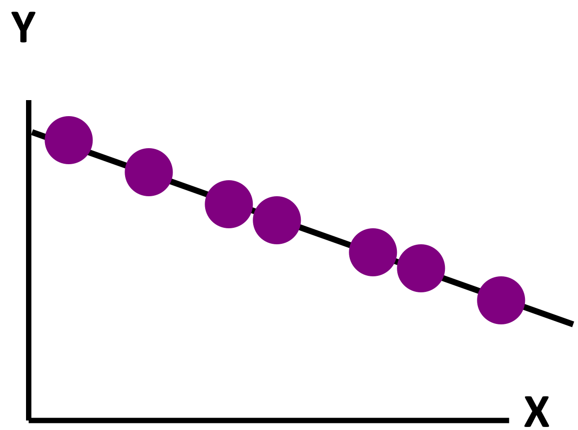
Coefficient of Determination, r^2

- The coefficient of determination is the portion of the total variation in the dependent variable that is explained by variation in the independent variable
- The coefficient of determination is also called r-squared and is denoted as r^2

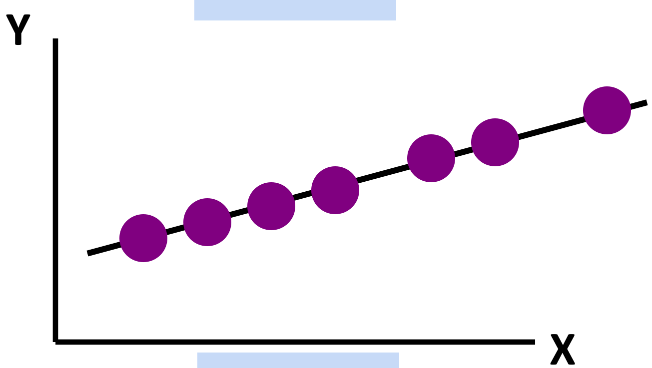
$$\boxed{r^2 = \frac{SSR}{SST} = \frac{\text{regression sum of squares}}{\text{total sum of squares}}}$$

note: $\boxed{0 \le r^2 \le 1}$

Examples of Approximate r^2 Values



$r^2 = 1$



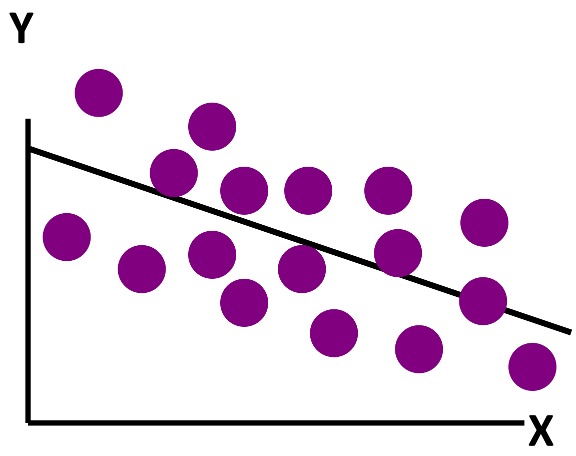
$r^2 = 1$

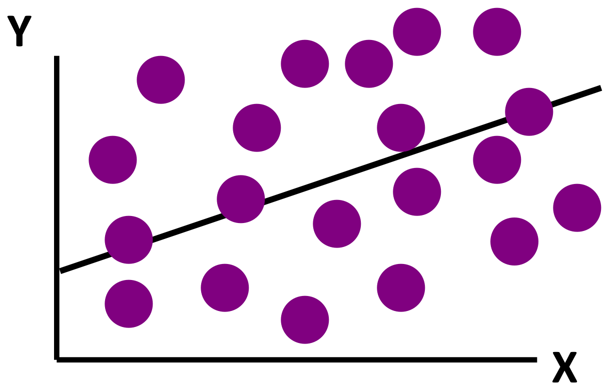
$r^2 = 1$

Perfect linear relationship between X and Y:

100% of the variation in Y is explained by variation in X

Examples of Approximate r^2 Values



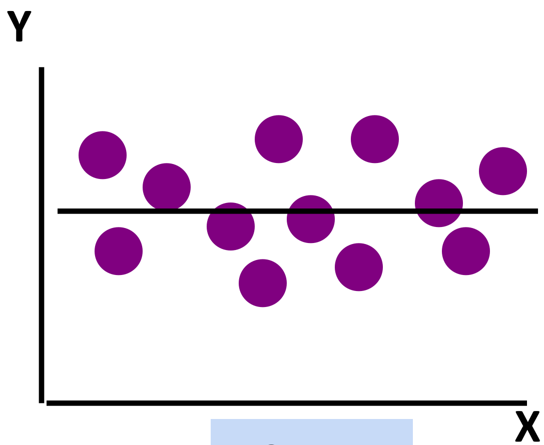


$0 < r^2 < 1$

Weaker linear relationships between X and Y:

Some but not all of the variation in Y is explained by variation in X

Examples of Approximate r^2 Values



$r^2 = 0$

$r^2 = 0$

No linear relationship between X and Y:

The value of Y does not depend on X. (None of the variation in Y is explained by variation in X)

Standard Error of Estimate

- The standard deviation of the variation of observations around the regression line is estimated by

$$S_{YX} = \sqrt{\frac{SSE}{n-2}} = \sqrt{\frac{\sum_{i=1}^{n}(Y_i - \hat{Y}_i)^2}{n-2}}$$

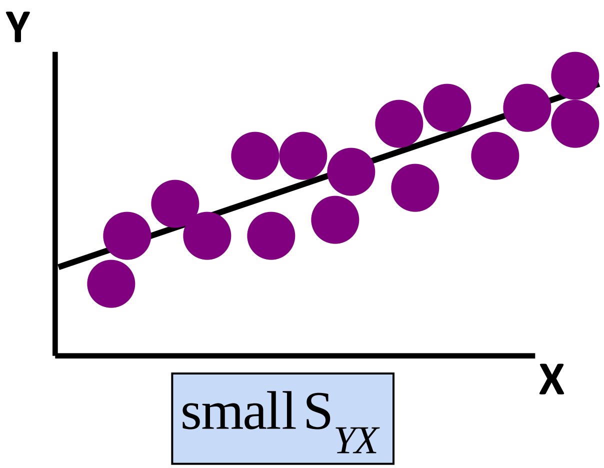
Where

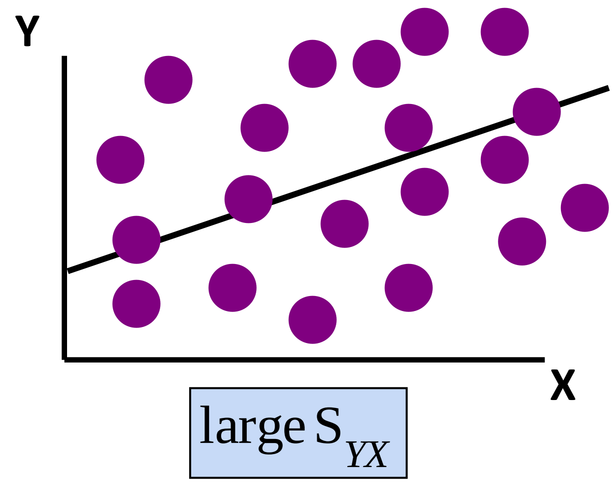
SSE = error sum of squares

n = sample size

Comparing Standard Errors

S_{YX} is a measure of the variation of observed Y values from the regression line





The magnitude of S_{YX} should always be judged relative to the size of the Y values in the sample data

i.e., S_{YX} = \$41.33K is moderately small relative to house prices in the \$200K - \$400K range

Assumptions of Regression
L.I.N.E

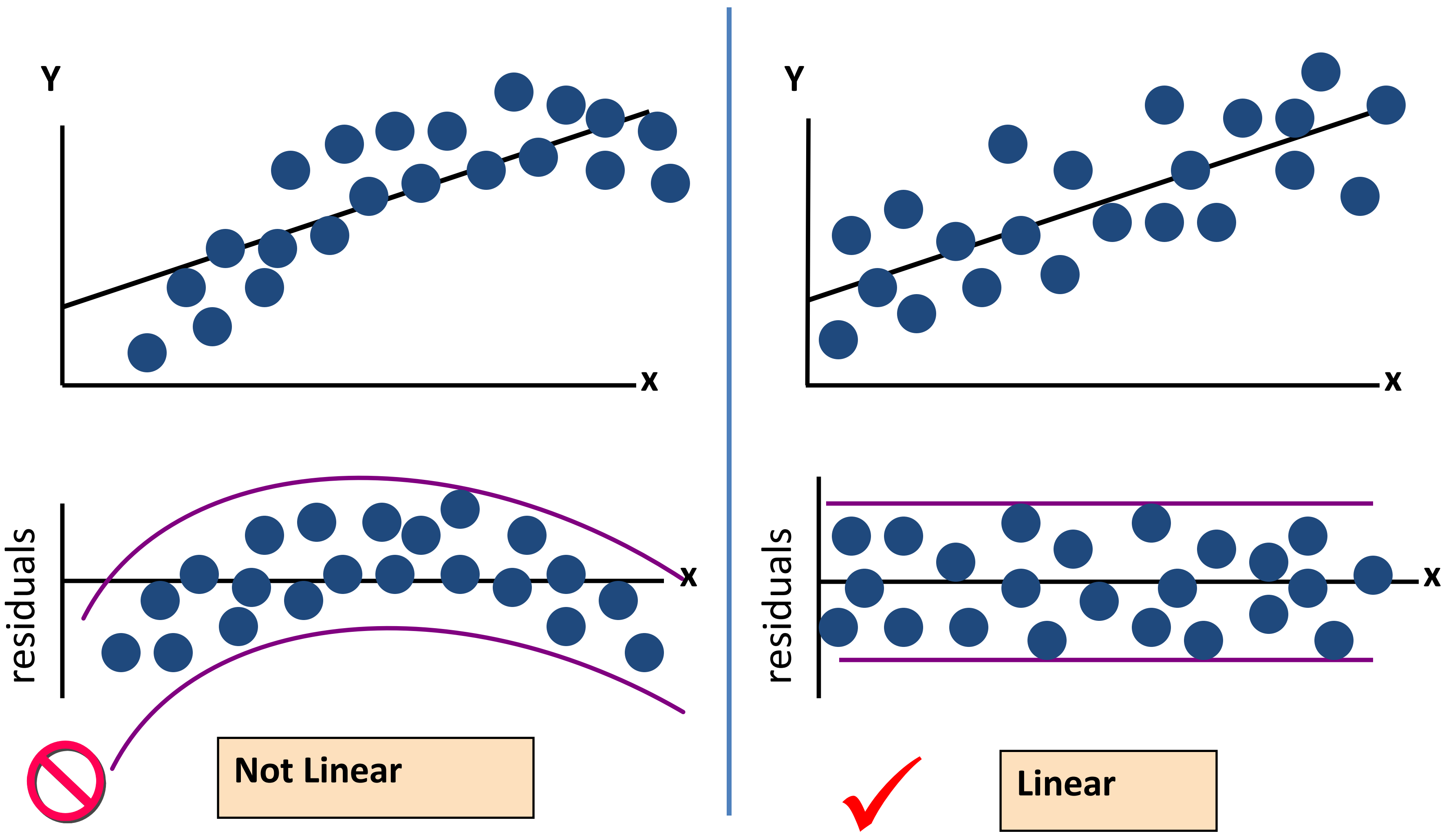
- Linearity
  - The relationship between X and Y is linear
- Independence of Errors
  - Error values are statistically independent
- Normality of Error
  - Error values are normally distributed for any given value of X
- Equal Variance (also called homoscedasticity)
  - The probability distribution of the errors has constant variance

Residual Analysis

$$\boxed{e_i = Y_i - \hat{Y}_i}$$

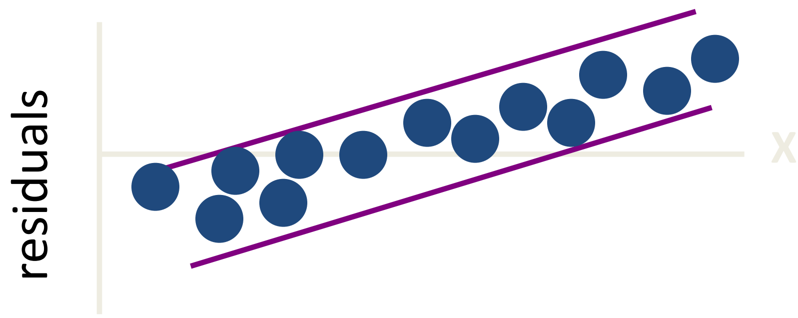
- The residual for observation i, e_i, is the difference between its observed and predicted value
- Check the assumptions of regression by examining the residuals
  - Examine for linearity assumption
  - Evaluate independence assumption
  - Evaluate normal distribution assumption
  - Examine for constant variance for all levels of X (homoscedasticity)
- Graphical Analysis of Residuals
  - Can plot residuals vs. X

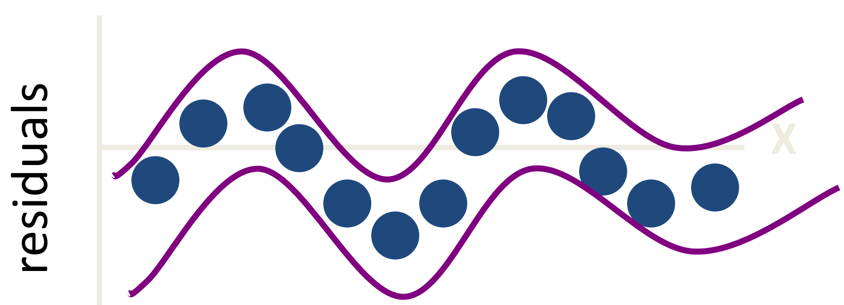
Residual Analysis for Linearity



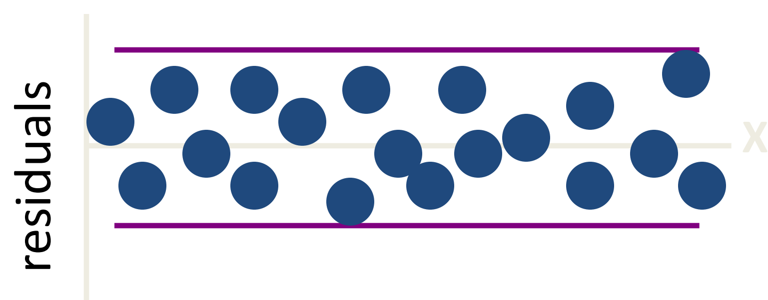
Residual Analysis for Independence

Not Independent





Independent

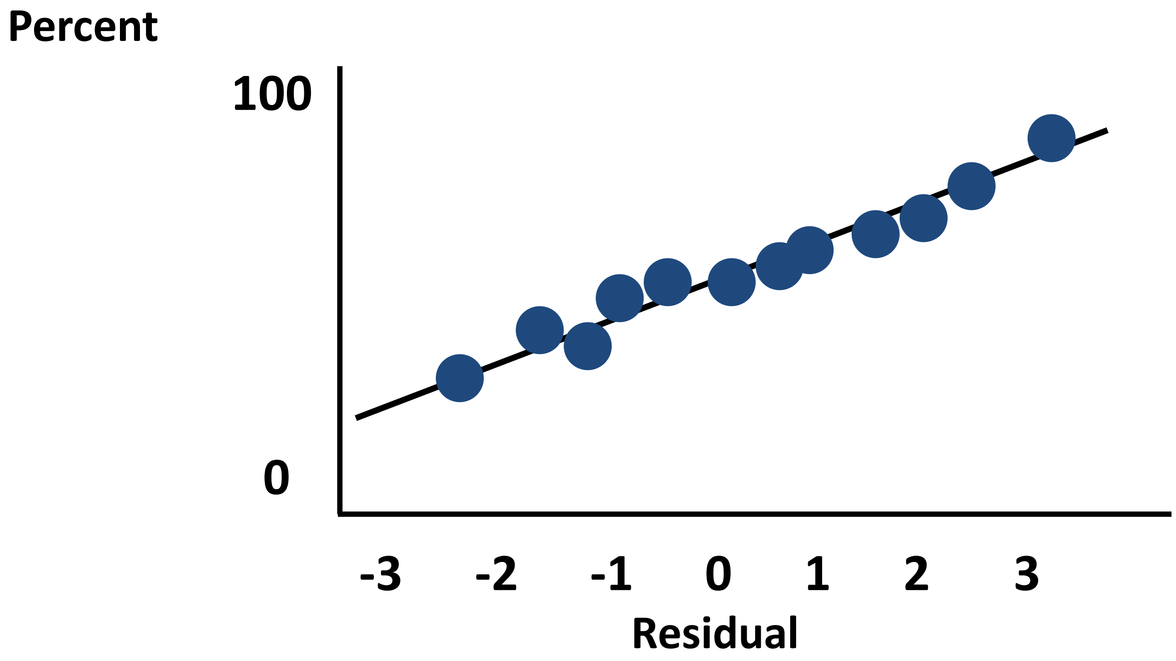


Checking for Normality

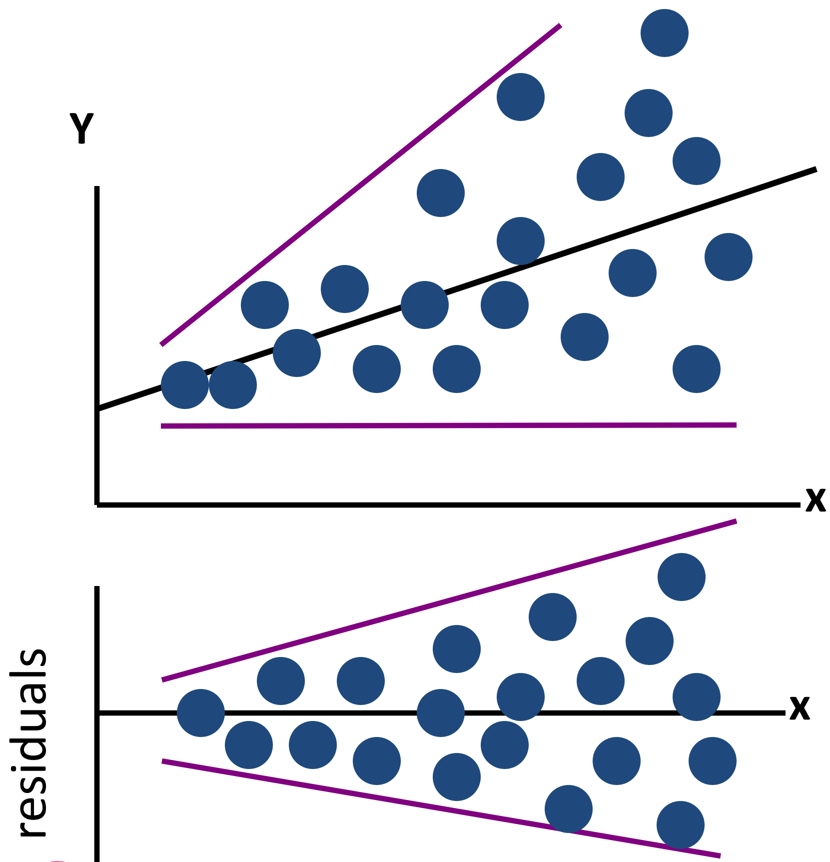
- Examine the Stem-and-Leaf Display of the Residuals
- Examine the Boxplot of the Residuals
- Examine the Histogram of the Residuals
- Construct a Normal Probability Plot of the Residuals

Residual Analysis for Normality

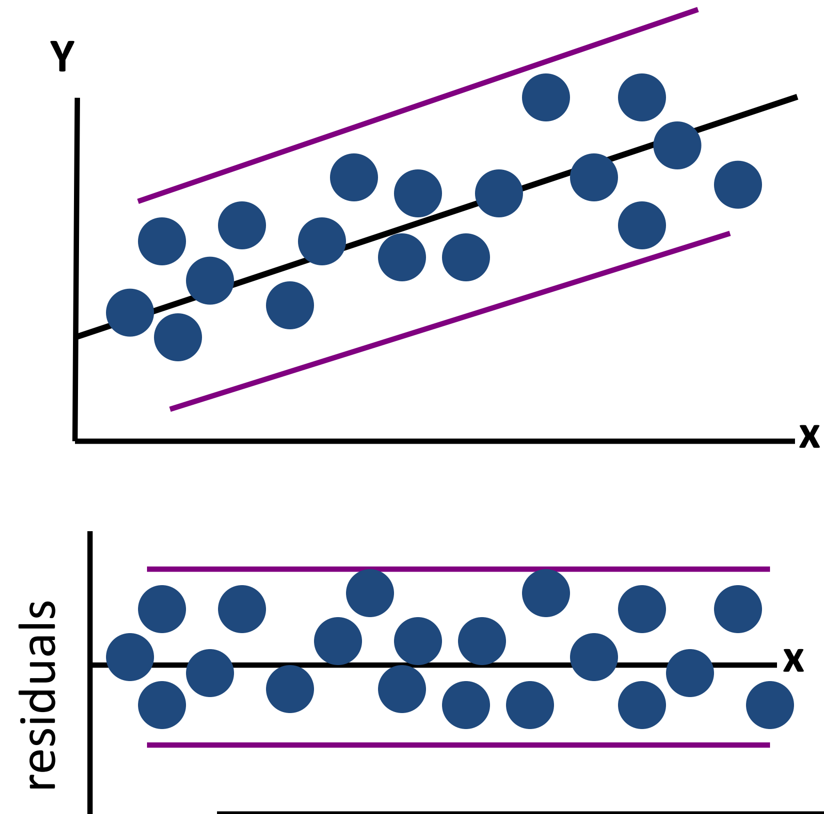
When using a normal probability plot, normal errors will approximately display in a straight line



Residual Analysis for Equal Variance



Non-constant variance



Constant variance

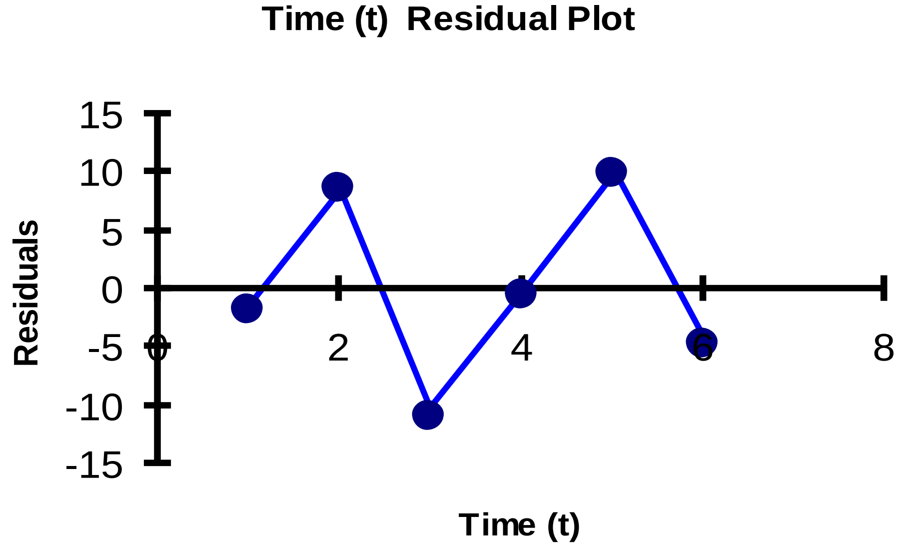
Measuring Autocorrelation: The Durbin-Watson Statistic

- Used when data are collected over time to detect if autocorrelation is present

- Autocorrelation exists if residuals in one time period are related to residuals in another period

Autocorrelation

- Autocorrelation is correlation of the errors (residuals) over time



- Here, residuals show a cyclic pattern (not random.) Cyclical patterns are a sign of positive autocorrelation

- Violates the regression assumption that residuals are random and independent

The Durbin-Watson Statistic

- The Durbin-Watson statistic is used to test for autocorrelation

H_0: residuals are not correlated

H_1: positive autocorrelation is present

$$D = \frac{\sum_{i=2}^{n} (e_i - e_{i-1})^2}{\sum_{i=1}^{n} e_i^2}$$

- The possible range is $0 \leq D \leq 4$
- D should be close to 2 if H_0 is true
- D less than 2 may signal positive autocorrelation, D greater than 2 may signal negative autocorrelation

Testing for Positive Autocorrelation

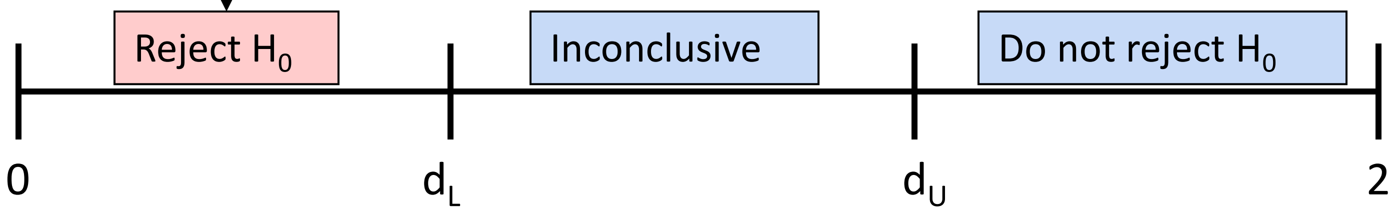
H_0: positive autocorrelation does not exist

H_1: positive autocorrelation is present

- Calculate the Durbin-Watson test statistic = D
  (The Durbin-Watson Statistic can be found using Excel or Minitab)

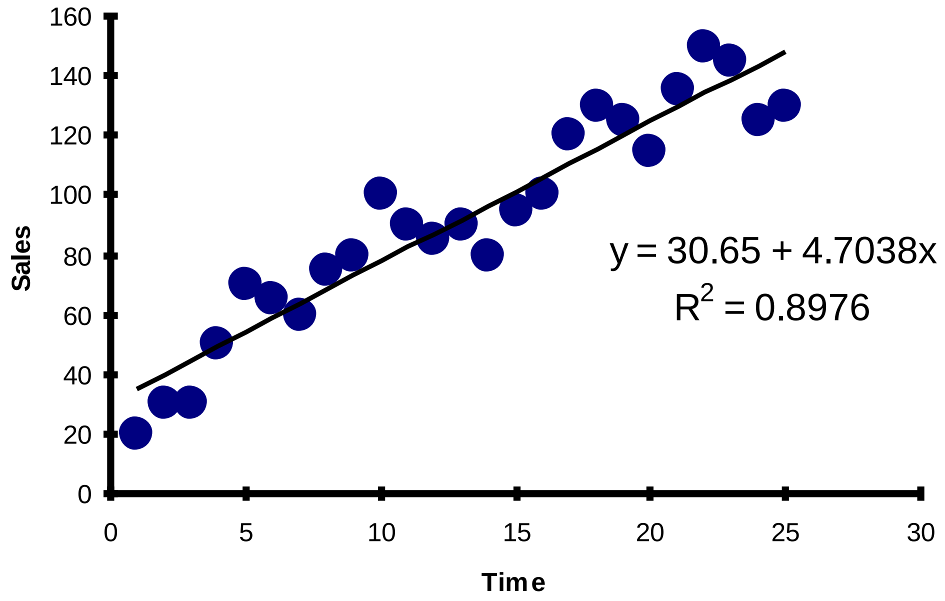
- Find the values d_L and d_U from the Durbin-Watson table
  (for sample size **n** and number of independent variables **k**)

Decision rule: reject H_0 if $D < d_L$



Testing for Positive Autocorrelation

- Suppose we have the following time series data:



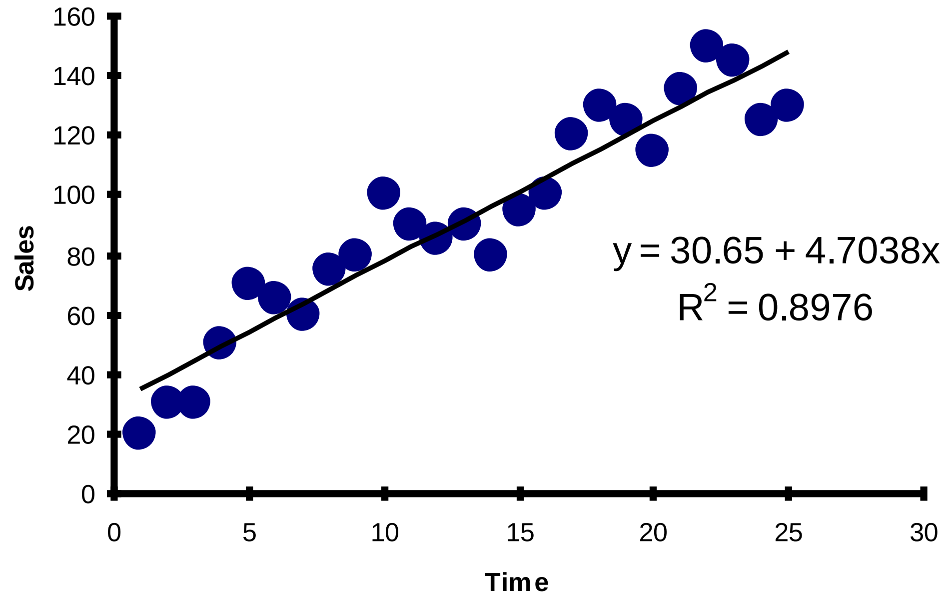
- Is there autocorrelation?

Testing for Positive Autocorrelation

- Example with $n = 25$:

Excel/PHStat output:

Durbin-Watson Calculations	
Sum of Squared Difference of Residuals	3296.18
Sum of Squared Residuals	3279.98
Durbin-Watson Statistic	**1.00494**



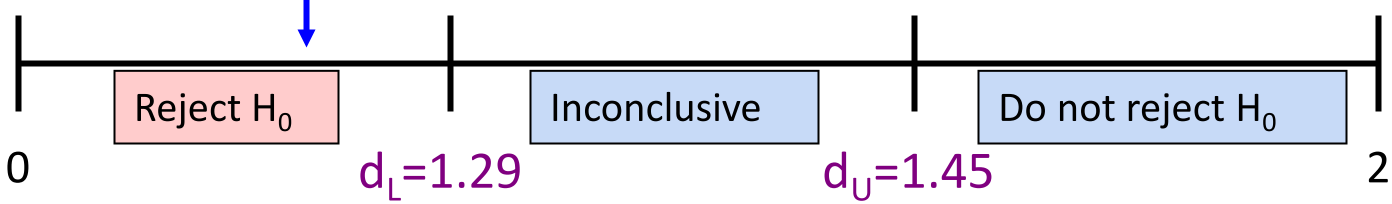
$$D = \frac{\sum_{i=2}^{n}(e_i - e_{i-1})^2}{\sum_{i=1}^{n} e_i^2} = \frac{3296.18}{3279.98} = 1.00494$$

Testing for Positive Autocorrelation

- Here, n = 25 and there is k = 1 one independent variable
- Using the Durbin-Watson table, $d_L = 1.29$ and $d_U = 1.45$
- $D = 1.00494 < d_L = 1.29$, so reject H_0 and conclude that significant positive autocorrelation exists

Decision: **reject H_0** since

$D = 1.00494 < d_L$



Inferences About the Slope

- The standard error of the regression slope coefficient (b_1) is estimated by

$$S_{b_1} = \frac{S_{YX}}{\sqrt{SSX}} = \frac{S_{YX}}{\sqrt{\sum (X_i - \overline{X})^2}}$$

where:

S_{b_1} = Estimate of the standard error of the slope

$S_{YX} = \sqrt{\frac{SSE}{n-2}}$ = Standard error of the estimate

Inferences About the Slope: t Test

- t test for a population slope
  - Is there a linear relationship between X and Y?
- Null and alternative hypotheses
  - H_0: $\beta_1 = 0$ (no linear relationship)
  - H_1: $\beta_1 \neq 0$ (linear relationship does exist)
- Test statistic

$$t_{STAT} = \frac{b_1 - \beta_1}{S_{b_1}}$$

$$d.f. = n - 2$$

where:

b_1 = regression slope coefficient

β_1 = hypothesized slope

S_{b1} = standard error of the slope

Inferences About the Slope: t Test Example

House Price in $1000s (y)	Square Feet (x)
245	1400
312	1600
279	1700
308	1875
199	1100
219	1550
405	2350
324	2450
319	1425
255	1700

Estimated Regression Equation:

$$\text{house price} = 98.25 + 0.1098\,(\text{sq.ft.})$$

The slope of this model is 0.1098

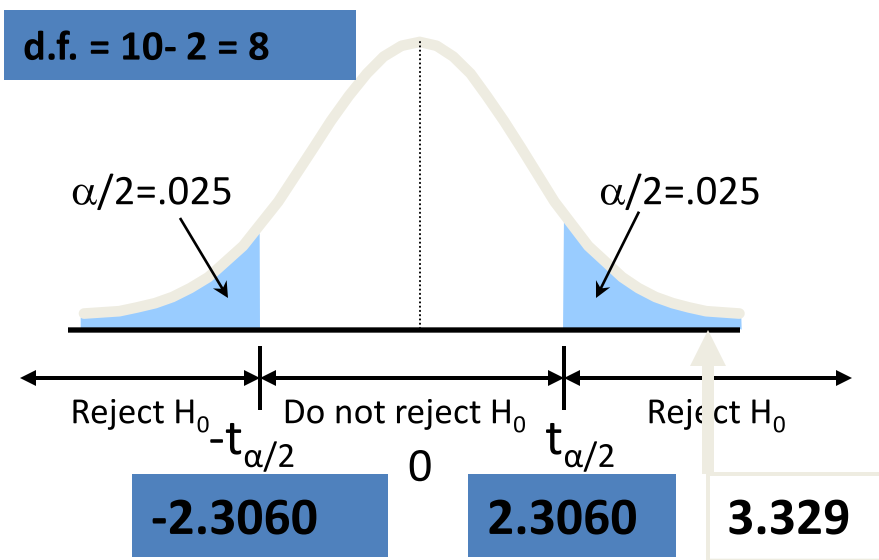
Is there a relationship between the square footage of the house and its sales price?

Inferences About the Slope: t Test Example

Test Statistic: $\mathbf{t_{STAT} = 3.329}$

H_0: $\beta_1 = 0$

H_1: $\beta_1 \neq 0$



Decision: Reject H_0

There is sufficient evidence that square footage affects house price

F Test for Significance

- F Test statistic:

$$F_{STAT} = \frac{MSR}{MSE}$$

where

$$MSR = \frac{SSR}{k}$$

$$MSE = \frac{SSE}{n-k-1}$$

where F_{STAT} follows an F distribution with k numerator and $(n - k - 1)$ denominator degrees of freedom

(k = the number of independent variables in the regression model)

F Test for Significance

$H_0: \beta_1 = 0$

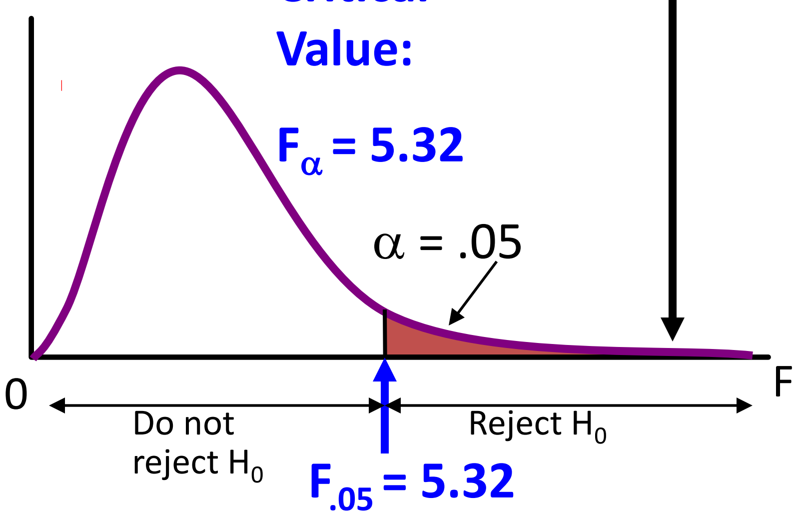
$H_1: \beta_1 \neq 0$

$\alpha = .05$

$\underbrace{df_1 = 1 \qquad df_2 = 8}$

Critical Value:

$F_\alpha = 5.32$



Test Statistic:

$$F_{STAT} = \frac{MSR}{MSE} = 11.08$$

Decision:

Reject H_0 at $\alpha = 0.05$

Conclusion:

There is sufficient evidence that house size affects selling price

t Test for a Correlation Coefficient

- Hypotheses

$H_0: \rho = 0$ (no correlation between X and Y)
$H_1: \rho \neq 0$ (correlation exists)

- Test statistic

$$t_{STAT} = \frac{r - \rho}{\sqrt{\dfrac{1 - r^2}{n - 2}}}$$

(with n – 2 degrees of freedom)

where

$$r = +\sqrt{r^2} \quad \text{if } b_1 > 0$$

$$r = -\sqrt{r^2} \quad \text{if } b_1 < 0$$

t-test For A Correlation Coefficient

Is there evidence of a linear relationship between square feet and house price at the .05 level of significance?

H_0: ρ = 0 (No correlation)

H_1: ρ ≠ 0 (correlation exists)

α =.05 , df = 10 - 2 = 8

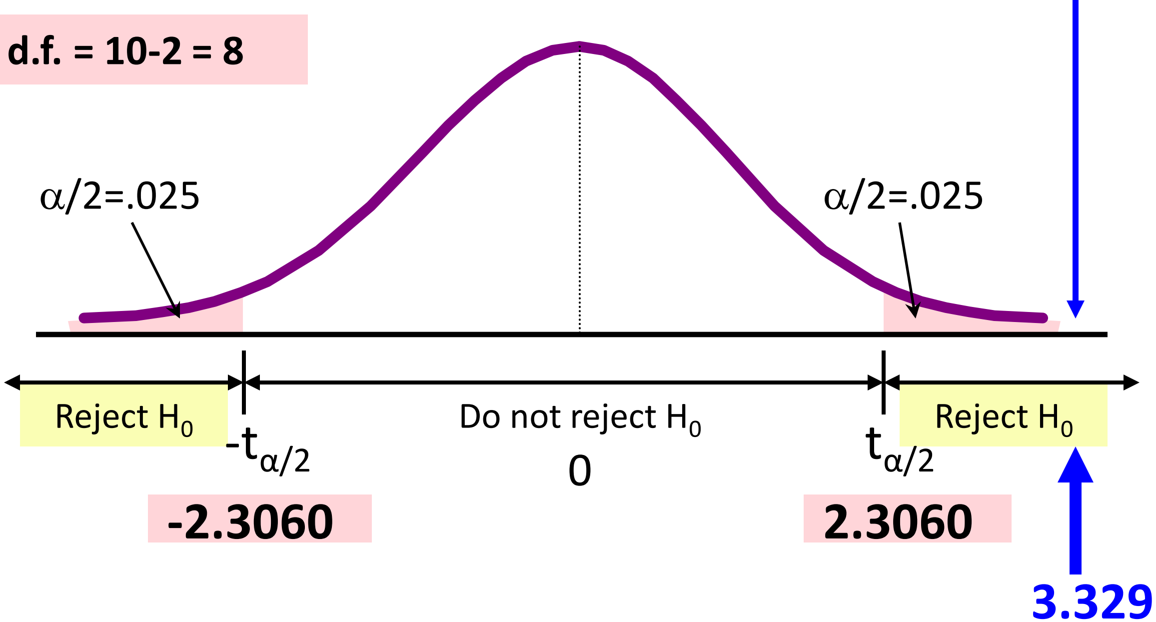
$$t_{STAT} = \frac{r - \rho}{\sqrt{\frac{1 - r^2}{n - 2}}} = \frac{.762 - 0}{\sqrt{\frac{1 - .762^2}{10 - 2}}} = 3.329$$

t-test For A Correlation Coefficient

$$t_{STAT} = \frac{r-\rho}{\sqrt{\frac{1-r^2}{n-2}}} = \frac{.762-0}{\sqrt{\frac{1-.762^2}{10-2}}} = 3.329$$

Decision:
Reject H_0

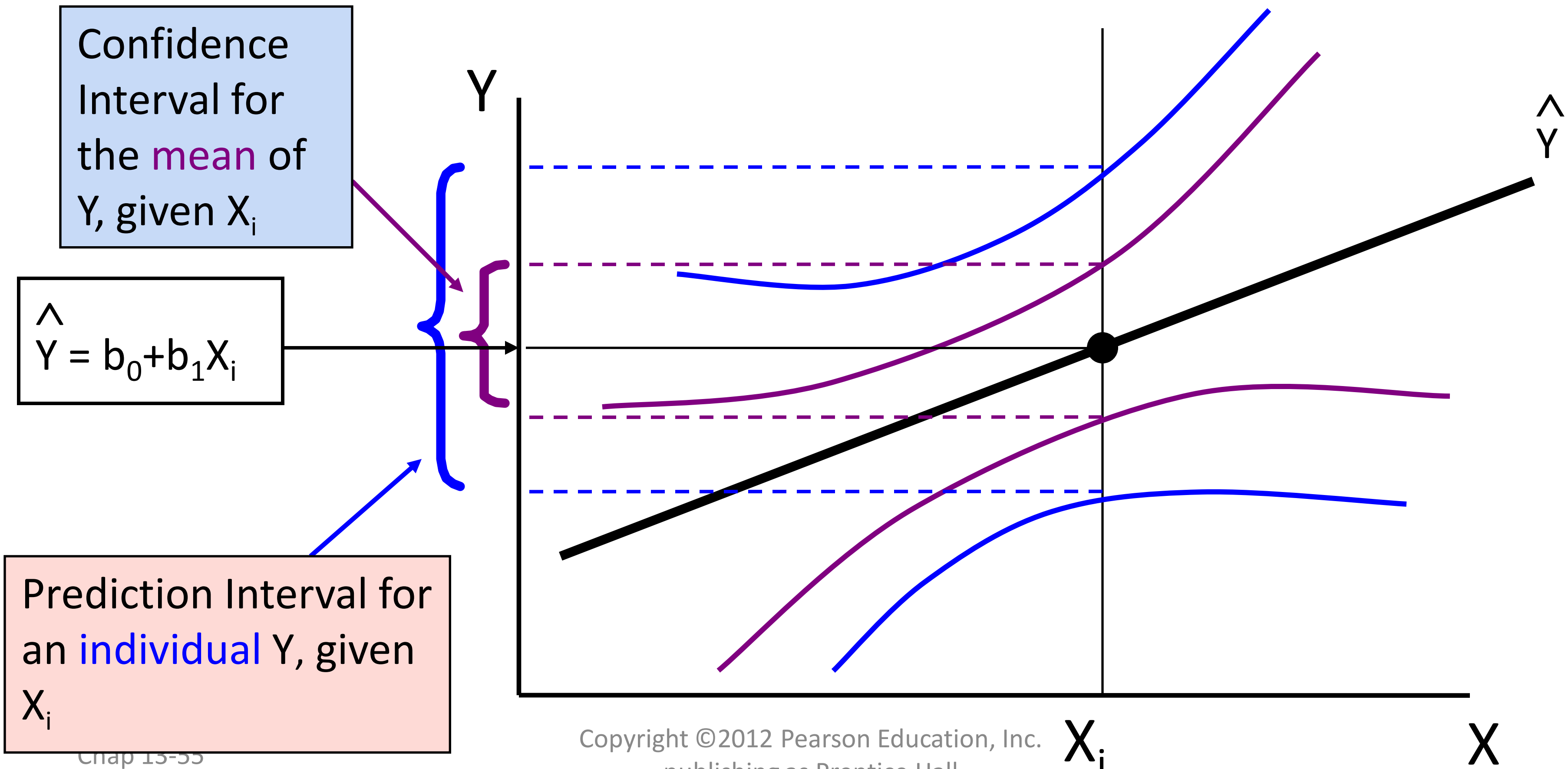
Conclusion:
There **is evidence** of a linear association at the 5% level of significance



Estimating Mean Values and Predicting Individual Values

DCOVA

Goal: Form intervals around Y to express uncertainty about the value of Y for a given X_i



Chap 13-55

Pitfalls of Regression Analysis

- Lacking an awareness of the assumptions underlying least-squares regression
- Not knowing how to evaluate the assumptions
- Not knowing the alternatives to least-squares regression if a particular assumption is violated
- Using a regression model without knowledge of the subject matter
- Extrapolating outside the relevant range

Strategies for Avoiding the Pitfalls of Regression

- Start with a scatter plot of X vs. Y to observe possible relationship
- Perform residual analysis to check the assumptions
  - Plot the residuals vs. X to check for violations of assumptions such as homoscedasticity
  - Use a histogram, stem-and-leaf display, boxplot, or normal probability plot of the residuals to uncover possible non-normality

Strategies for Avoiding the Pitfalls of Regression

- If there is violation of any assumption, use alternative methods or models
- If there is no evidence of assumption violation, then test for the significance of the regression coefficients and construct confidence intervals and prediction intervals
- Avoid making predictions or forecasts outside the relevant range

EXERCISE

13.4 (cont’d)

Seorang manajer marketing dari sebuah supermarket ingin menggunakan tinggi rak untuk memprediksi penjualan makanan binatang peliharaan. Sampel acak yang terdiri dari 12 toko yang ukurannya hampir sama dipilih dengan hasil sebagai berikut:

13.4 (cont’d)

Toko	Tinggi Rak (ft)	Penjualan mingguan ($)
1	5	160
2	5	220
3	5	140
4	10	190
5	10	240
6	10	260
7	15	230
8	15	270
9	15	280
10	20	260
11	20	290
12	20	310

13.4

a. Gunakan metode kuadrat terkecil untuk menentukan koefisien regresi b_0 dan b_1
b. Interpretasikan arti b_0 dan b_1 pada permasalahan ini.
c. Prediksikan penjualan mingguan makanan binatang peliharaan untuk toko dengan tinggi rak 8 ft.

Example

Hours Spent Studying (X)	Math SAT Score (Y)
4	390
9	580
10	650
14	730
4	410
7	530
12	600
22	790
1	350
3	400
8	590
11	640
5	450
6	520
10	690

THANK YOU