Decision Making Theory

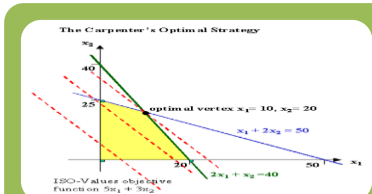
Week 4

Linear Programming

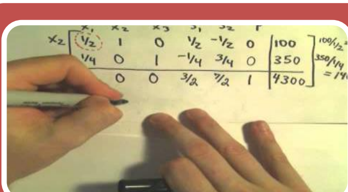
Simplex Method - Minimize

Outlines

- Linear Programing Model – Problem Formulation



- Linear Programing Model – Graphic Solution



- Linear Programing Model – Simplex Method (Maximize)

- Linear Programing Model – Simplex Method (Minimize and Non-Standard)

Linear Programming Problems

Standard Maximization

- Objective: Maximize
- Constraints: ALL with ≤
- Standard simplex

Standard Minimization

- Objective: Minimize
- Constraints: All with ≥
- Dual

Non-Standard

- Objective: Max or Min
- Constraints: Mixed, can be ≤, ≥ or =
- Two Phase

STANDARD MINIMIZATION PROBLEM

Holiday Meal Turkey Ranch

Minimize cost = $2X_1 + 3X_2$

subject to

$$5X_1 + 10X_2 \geq 90$$

$$4X_1 + 3X_2 \geq 48$$

$$0.5 X_1 \geq 1.5$$

$$X_1, X_2 \geq 0$$

The Dual

Minimize cost = $2X_1 + 3X_2$

subject to

$5X_1 + 10X_2 \geq 90$

$4X_1 + 3X_2 \geq 48$

$0.5\,X_1 \geq 1.5$

$X_1, X_2 \geq 0$

$$A = \left[\begin{array}{cc|c} 5 & 10 & 90 \\ 4 & 3 & 48 \\ 0.5 & 0 & 1.5 \\ \hline 2 & 3 & 0 \end{array}\right]$$

$$A^T = \left[\begin{array}{ccc|c} 5 & 4 & 0.5 & 2 \\ 10 & 3 & 0 & 3 \\ \hline 90 & 48 & 1.5 & 0 \end{array}\right]$$

The Dual

$$A^T = \left[\begin{array}{ccc|c} 5 & 4 & 0.5 & 2 \\ 10 & 3 & 0 & 3 \\ \hline 90 & 48 & 1.5 & 0 \end{array}\right]$$

Maximize

$$Z = 90Y_1 + 48Y_2 + 1.5Y_3$$

subject to

$$5Y_1 + 4Y_2 + 0.5Y_3 \leq 2$$

$$10Y_1 + 3Y_2 \leq 3$$

$$Y_1, Y_2, Y_3 \geq 0$$

Continue with the simplex method to solve standard maximization problems!!

Slack Variables

When using the dual, slack variables are variables in the former problem formulation. Therefore, slack variables in this problem are X_1 and X_2.

Constraints:

$$5Y_1 + 4Y_2 + 0.5Y_3 \le 2 \quad \Rightarrow \quad 5Y_1 + 4Y_2 + 0.5Y_3 + X_1 \le 2$$

$$10Y_1 + 3Y_2 \le 3 \quad \Rightarrow \quad 10Y_1 + 3Y_2 + X_2 \le 3$$

$$Z = 90Y_1 + 48Y_2 + 1.5Y_3 \qquad -90Y_1 - 48Y_2 - 1.5Y_3 + Z = 0$$

First Simplex Tableu

$5Y_1 + 4Y_2 + 0.5Y_3 + X_1 \leq 2 \rightarrow$ row 1

Basic Variable	Y_1	Y_2	Y_3	X_1	X_2	Z	RHS
X_1	5	4	0.5	1	0	0	2
X_2	10	3	0	0	1	0	3
Z	-90	-48	-1.5	0	0	1	0

$10Y_1 + 3Y_2 + X_2 \leq 3 \rightarrow$ row 2

$-90Y_1 - 48Y_2 - 1.5Y_3 + Z = 0 \rightarrow$ row 3

Simplex Tableu

Change the pivot number to 1 and other number in the same column to 0

Pivot Number

Basic Variable	Y_1	Y_2	Y_3	X_1	X_2	Z	RHS	
X_1	5	4	0.5	1	0	0	2	= 2/5 = 0.4
X_2	10	3	0	0	1	0	3	= 3/20 = 0.3
Z	-90	-48	-1.5	0	0	1	0	

Most negative, therefore variable Y_1 enter the solution mix

Smallest result of the ratio, therefore variable X_2 leave the solution mix

Second Simplex Tableu

Basic Var.	Y_1	Y_2	Y_3	X_1	X_2	Z	RHS	
X_1	5	4	0.5	1	0	0	2	= Row 2 * (-5/10) + Row 1
X_2	10	3	0	0	1	0	3	= Row 2 / 10
Z	-90	-48	-1.5	0	0	1	0	= Row 2 * (90/10) + Row 3

Basic Var.	Y_1	Y_2	Y_3	X_1	X_2	Z	RHS
X_1	0	2.5	0.5	1	-1	0	0.5
Y_1	1	0.3	0	0	0.1	0	0.3
Z	0	-21	-1.5	0	9	1	27

Row P still contain negative value

Second Simplex Tableu

Change the pivot number to 1 and other number in the same column to 0

Pivot Number

Basic Var.	Y_1	Y_2	Y_3	X_1	X_2	Z	RHS
X_1	0	2.5	0.5	1	-1	0	0.5
Y_1	1	0.3	0	0	0.1	0	0.3
Z	0	-21	-1.5	0	9	1	27

0.5/2.5 = 0.2

0.3/0.3 = 1

Most negative, therefore variable Y_2 enter the solution mix

Smallest result of the ratio, therefore variable X_1 leave the solution mix

Third Simplex Tableu

Basic Var.	Y_1	Y_2	Y_3	X_1	X_2	Z	RHS	
X_1	0	2.5	0.5	1	-1	0	0.5	= Row 1 / 2.5
Y_1	1	0.3	0	0	0.1	0	0.3	= Row 1 * (-0.3/2.5) + Row 2
Z	0	-21	-1.5	0	9	1	27	= Row 1 * (21/2.5) + Row 3

Basic Var.	Y_1	Y_2	Y_3	X_1	X_2	Z	RHS
Y_2	0	1	0.2	0.4	-0.2	0	0.2
Y_1	1	0	-0.06	-0.12	0.16	0	0.24
Z	0	0	2.7	8.4	4.8	1	31.2

OPTIMAL!!

Optimal Solution

Because this solution is the solution of the dual, use row Z and column X_1, X_2 as the optimal solution.

Basic Var.	Y_1	Y_2	Y_3	X_1	X_2	Z	RHS
Y_2	0	1	0.2	0.4	-0.2	0	0.2
Y_1	1	0	-0.06	-0.12	0.16	0	0.24
Z	0	0	2.7	8.4	4.8	1	31.2

$X_1 = 8.4$
$X_2 = 4.8$

Minimal cost = 31.2

EXERCISE

M7-22

Solve the following LP problem first graphically and then by the simplex algorithm:

Minimize cost = $4X_1 + 5X_2$

subject to $X_1 + 2X_2 \geq 80$

$3X_1 + X_2 \geq 75$

$X_1, X_2 \geq 0$

Objective function:

minimize $w = 3x_1 + 3x_2$

constraints:

$2x_1 + x_2 \geq 4$

$x_1 + 2x_1 \geq 4$

$x_1, x_2 \geq 0$

NON-STANDARD LINEAR PROGRAMMING PROBLEM

The Muddy River Chemical Company

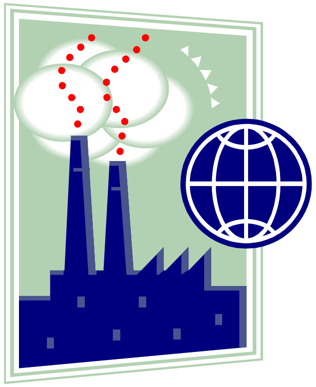
Minimize cost = $\$5X_1 + \$6X_2$

subject to $X_1 + X_2 = 1000$ lb

$X_1 \leq 300$ lb

$X_2 \geq 150$ lb

$X_1, X_2 \geq 0$



Surplus Variable

Greater-than-or-equal-to (≥) constraints involve the subtraction of a ***surplus variable*** rather than the addition of a slack variable.

Surplus is sometimes simply called

negative slack

$X_2 \geq 150$ → $X_2 - S_2 = 150$

Artificial Variable

Artificial variables are usually added to **greater-than-or-equal-to (≥)** constraints to avoid violating the non-negativity constraint.

When a constraint is already an **equality (=)**, artificial variable is also used to provide an automatic initial solution

Artificial Variable

- Equality

  $X_1 + X_2 = 1000$

- Greater than or equal

  $X_2 \geq 150$

- Equality

  $X_1 + X_2 + A_1 = 1000$

- Greater than or equal

  $X_2 - S_2 + A_2 = 150$

Before the final simplex solution has been reached, all artificial variables must be gone from the solution mix

Two Phase Method

Phase I

- **Objective:** Minimize all artificial variables

Phase II

- **Objective:** LP original objective

Conversions

Constraints:

$X_1 + X_2 = 1000$

$X_1 \leq 300$

$X_2 \geq 150$

$X_1 + X_2 + A_1 = 1000$

$X_1 + S_1 = 300$

$X_2 - S_2 + A_2 = 150$

Objective function:

Minimize $C = 5X_1 + 6X_2$

Phase I :

Minimize $Z = A_1 + A_2$

Phase II:

Minimize $C = 5X_1 + 6X_2$

Converting The Objective Function

In minimization problem, minimizing the cost objective is the same as ***maximizing the negative of the cost*** objective function

$$\text{Minimize } C = 5X_1 + 6X_2 \text{ and } Z = A_1 + A_2$$

$$\downarrow$$

$$\text{Maximize } (-C) = -5X_1 - 6X_2 \text{ and } (-Z) = -A_1 - A_2$$

Conversions

Constraints:

$X_1 + X_2 = 1000$	$X_1 + X_2 + A_1 = 1000$
$X_1 \leq 300$	$X_1 + S_1 = 300$
$X_2 \geq 150$	$X_2 - S_2 + A_2 = 150$

Objective function:

Phase I : $(-Z) = -A_1 - A_2$	Phase I : $A_1 + A_2 - Z = 0$
Phase II:	Phase II:
$(-C) = -5X_1 - 6X_2$	$5X_1 + 6X_2 - C = 0$

Simplex Tableu

$X_1 + X_2 + A_1 = 1000 \rightarrow$ row 1

$X_1 + S_1 \leq 300 \rightarrow$ row 2

$X_2 - S_2 + A_2 \geq 150 \rightarrow$ row 3

Basic Var.	X_1	X_2	S_1	S_2	A_1	A_2	Z	C	RHS
A_1	1	1	0	0	1	0	0	0	1000
S_1	1	0	1	0	0	0	0	0	300
A_2	0	1	0	-1	0	1	0	0	150
Z	0	0	0	0	1	1	(-1)	0	0
C	5	6	0	0	0	0	0	(-1)	0

$A_1 + A_2 - Z = 0 \rightarrow$ row 4

$5X_1 + 6X_2 - C = 0 \rightarrow$ row 5

Simplex Tableu

A1 and A2 are supposed to be basic variable. However, in this tableu the condition is not satisfy. → infeasible

Basic Var.	X_1	X_2	S_1	S_2	A_1	A_2	Z	C	RHS
A_1	1	1	0	0	1	0	0	0	1000
S_1	1	0	1	0	0	0	0	0	300
A_2	0	1	0	-1	0	1	0	0	150
Z	0	0	0	0	1	1	(-1)	0	0
C	5	6	0	0	0	0	0	(-1)	0

row 1 * (-1) + row 3 * (-1) + row 4

These values need to be converted to zero (0)

Phase I: First Simplex Tableu

Basic Variable condition is satisfied

Basic Var.	X_1	X_2	S_1	S_2	A_1	A_2	Z	C	RHS
A_1	1	1	0	0	1	0	0	0	1000
S_1	1	0	1	0	0	0	0	0	300
A_2	0	1	0	-1	0	1	0	0	150
Z	-1	-2	0	1	0	0	(-1)	0	-1150
C	5	6	0	0	0	0	0	(-1)	0

Continue with simplex procedure to eliminate the negative values in row Z first (**Phase I**)

Phase I: Second Simplex Tableu

Change the pivot number to 1 and other number in the same column to 0

Pivot Number

Basic Var.	X_1	X_2	S_1	S_2	A_1	A_2	Z	C	RHS
A_1	1	1	0	0	1	0			1000
S_1	1	0	1	0	0	0			300
A_2	0	1	0	-1	0	1			150
Z	-1	**-2**	0	1	0	0	(-1)	0	-1150
C	5	6	0	0	0	0	0	(-1)	0

1000/1 = 1000

300/0 = ~

150/1 = 150

Most negative, therefore variable X_2 enter the solution mix

Smallest result of the ratio, therefore variable A_2 leave the solution mix

Phase I: Second Simplex Tableu

Basic Var.	X_1	X_2	S_1	S_2	A_1	A_2	Z	C	RHS
A_1	1	1	0	0	1				1000
S_1	1	0	1	0	0				300
A_2	0	1	0	-1	0				150
Z	-1	**-2**	0	1	0				1150
C	5	6	0	0	0				0

= row 3 * (-1) + row 1

= row 2

= row 3

= row 3 * 2 + row 4

= row 3 * (-6) + row 5

Basic Var.	X_1	X_2	S_1	S_2	A_1	A_2	Z	C	RHS
A_1	1	0	0	1	1	-1	0	0	850
S_1	1	0	1	0	0	0	0	0	300
X_2	0	1	0	-1	0	1	0	0	150
Z	-1	0	0	-1	0	2	(-1)	0	-850
C	5	0	0	6	0	-6	0	(-1)	-900

Row Z still contain negative values

Phase I: Third Simplex Tableu

Change the pivot number to 1 and other number in the same column to 0

Pivot Number

Basic Var.	X_1	X_2	S_1	S_2	A_1	A_2	Z	C	RHS
A_1	1	0	0	1	1	-1			850
S_1	1	0	1	0	0	0			300
X_2	0	1	0	-1	0	1			150
Z	-1	0	0	-1	0	2	(-1)	0	-850
C	5	0	0	6	0	-6	0	(-1)	-900

850/1 = 850

300/0 = ~

-

Most negative, therefore variable S_2 enter the solution mix

Smallest result of the ratio, therefore variable A_1 leave the solution mix

Phase I: Third Simplex Tableu

Basic Var.	X_1	X_2	S_1	S_2	A_1	A_2	Z	C	RHS
A_1	1	0	0	1	1	[illegible]	[illegible]	[illegible]	850
S_1	1	0	1	0	0	[illegible]	[illegible]	[illegible]	300
X_2	0	1	0	-1	0	[illegible]	[illegible]	[illegible]	150
Z	-1	0	0	-1	0	[illegible]	[illegible]	[illegible]	-850
C	5	0	0	6	0	[illegible]	[illegible]	[illegible]	-900

= row 1

= row 2

= row 1 + row 3

= row 1 + row 4

= row 1 * (-6) + row 5

Basic Var.	X_1	X_2	S_1	S_2	A_1	A_2	Z	C	RHS
S_2	1	0	0	1	1	-1	0	0	850
S_1	1	0	1	0	0	0	0	0	300
X_2	1	1	0	0	1	0	0	0	1000
Z	0	0	0	0	1	1	(-1)	0	0
C	1	0	0	0	6	0	0	(-1)	-6000

Phase I OPTIMAL!!

Phase II: First Simplex Tableu

Change the pivot number to 1 and other number in the same column to 0

Pivot Number

Basic Var.	X_1	X_2	S_1	S_2	C	RHS
S_2	1	0	0	1	0	850
S_1	1	0	1	0	0	300
X_2	1	1	0	0	0	1000
C	-1	0	0	0	(-1)	-6000

850/1 = 850

300/1 = 300

1000/1 = 1000

Most negative, therefore variable X_1 enter the solution mix

Smallest result of the ratio, therefore variable S_1 leave the solution mix

Third Simplex Tableu

Basic Var.	X_1	X_2	S_1	S_2	C	RHS	
S_2	1	0	0	1	0	850	= Row 2 * (-1) + row 1
S_1	1	0	1	0	0	300	= Row 2
X_2	1	1	0	0	0	1000	= Row 2 * (-1) + Row 3
C	-1	0	0	0	(-1)	-6000	= Row 2 + Row 4

Basic Var.	X_1	X_2	S_1	S_2	C	RHS
S_2	0	0	-1	1	0	550
X_1	1	0	1	0	0	300
X_2	0	1	-1	0	0	700
C	0	0	1	0	(-1)	-5700

OPTIMAL!!

Optimal Solution

Remember that the objective function was converted. Thus:

$$\text{Max } (-C) = -5700 \rightarrow \text{Min } C = 5700$$

Basic Var.	X_1	X_2	S_1	S_2	C	RHS
S_2	0	0	-1	1	0	550
X_1	1	0	1	0	0	300
X_2	0	1	-1	0	0	700
C	0	0	1	0	(-1)	-5700

$X_1 = 300$

$X_2 = 700$

EXERCISE

M7-27

A Pharmaceutical firm is about to begin production of three new drugs. An objective function designed to minimize ingredients costs and three production constraints are as follows:

Minimize cost = $50X_1 + 10X_2 + 75X_3$

subject to

$$X_1 - X_2 = 1.000$$

$$2X_2 + 2X_3 = 2.000$$

$$X_1 \le 1.500$$

$$X_1, X_2, X_3 \ge 0$$

SPECIAL CASE IN SIMPLEX METHOD

1. Infeasibility

No feasible solution is possible if an artificial variable remains in the solution mix

Basic Var.	X_1	X_2	S_1	S_2	A_1	A_2	Z	C	RHS
X_1	1	0	-2	3	-1	0	0	0	200
X_2	0	1	1	2	-2	0	0	0	100
A_2	0	0	0	-1	-1	1	0	0	20
Z	0	0	2	3	0	0	(-1)	0	-850
C	0	0	0	-31	21	0	0	(-1)	-900

Phase I already optimal (no negative value)

2. Unbounded Solutions

Unboundedness describes linear programs that do not have finite solutions

Basic Var.	X_1	X_2	S_1	S_2	**P**	**RHS**
X_2	-1	1	2	0	0	30
S_2	-2	0	-1	1	0	10
P	-15	0	18	0	1	270

Since both pivot column numbers are negative, an unbounded solution is indicated

3. Degeneracy

Degeneracy develops when three constraints pass through a single point

Basic Var.	X_1	X_2	X_3	S_1	S_2	S_3	C	RHS
X_2	0.25	1	1	[illegible]	[illegible]	[illegible]	0	10
S_2	4	0	0.33	[illegible]	[illegible]	[illegible]	0	20
S_3	2	0	2	[illegible]	[illegible]	[illegible]	0	10
C	-3	0	6	16	0	0	1	80

10/0.25 = 40

20/4 = 5

10/2 = 5

Tie for the smallest ratio indicates degeneracy

Degeneracy could lead to a situation known as ***cycling***, in which the simplex algorithm alternates back and forth between the same non-optimal solutions

4. More Than One Optimal Solution

If the value of P (objective function's row) is equal to 0 for a variable that is not in the solution mix, more than one optimal solution exists.

Basic Var.	**X_1**	**X_2**	**S_1**	**S_2**	**P**	**RHS**
X_2	1.5	1	1	0	0	6
S_2	1	0	0.5	1	0	3
P	0	0	2	0	1	12

HOMEWORK

1

John's Locomotive Works manufactures a model locomotive. It comes in two versions--a standard (X1), and a deluxe (X2). The standard version generates $250 per locomotive for the standard version, and $350 per locomotive for the deluxe version. One constraint on John's production is labor hours. He only has 40 hours per week for assembly. The standard version requires 250 minutes each, while the deluxe requires 350 minutes.

1

John's milling machine is also a limitation. There are only 20 hours a week available for the milling machine. The standard unit requires 60 minutes, while the deluxe requires 120. Formulate as a linear programming problem, and solve using either the graphical or corner points solution method.

2

The Queen City Nursery manufactures bags of potting soil from compost and topsoil. Each cubic foot of compost costs 12 cents and contains 4 pounds of sand, 3 pounds of clay, and 5 pounds of humus. Each cubic foot of topsoil costs 20 cents and contains 3 pounds of sand, 6 pounds of clay, and 12 pounds of humus. Each bag of potting soil must contain at least 12 pounds of sand, 12 pounds of clay, and 10 pounds of humus.

2

Plot the constraints and identify the feasible region. Graphically or with corner points find the best combination of compost and topsoil that meets the stated conditions at the lowest cost per bag. Identify the lowest cost possible

3

What combination of x and y will yield the optimum for this problem?

Maximize \$3x + \$15y,

subject to

(1) $2x + 4y \leq 12$ and

(2) $5x + 2y \leq 10$.

4

What combination of x_1 and x_2 will yield the optimum for this problem?

Minimize $C = 2x_1 + x_2$

subject to

$5x_1 + x_2 \geq 9$,

$2x_1 + 2x_2 \leq 10$, and

$x_1, x_2 \geq 0$

THANK YOU